8 Half infinite string, boundary conditions, and the reflection method

8.1 Half-infinite string

In this lecture I will assume that my string is half-infinite, that is, in addition to the initial conditions on $x > 0$ I also have a boundary condition at the point $x = 0$. When I derived the wave equation as the limit of the system of masses connected by springs, I showed that the boundary condition of the form

$$u(t, 0) = 0$$

corresponds physically to the fixed end of the string. It can be argued that the boundary condition

$$u_x(t, 0) = 0$$

corresponds to a free end of the string. Let me start first with the fixed end.

I consider the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad t > 0,\, x > 0, \\ u(0, x) &= f(x), \quad x > 0, \\ u_t(0, x) &= g(x), \quad x > 0, \\ u(t, 0) &= 0, \quad t > 0. \end{aligned} \tag{8.1}$$

To solve problem (8.1) I will prove the following lemma.

Lemma 8.1. *Let the initial conditions for the wave equation on the infinite string are odd functions with respect to some point x_0 then the corresponding solution at this point is equal to zero.*

Proof. Without loss of generality, assume that $x_0 = 0$. Then I have that $f(x) = -f(-x)$, $g(x) = -g(-x)$. Using d'Alembert's formula for $x = 0$ yields

$$u(t, 0) = \frac{f(-ct) + f(ct)}{2} + \frac{1}{2c} \int_{-ct}^{ct} g(s) \mathrm{d}s = 0,$$

since the first term is zero because f is odd and the second one is also zero as an integral of an odd function through a symmetric interval. ■

This lemma implies that to solve (8.1) all I need is to extend my initial conditions in an odd fashion and apply d'Alembert's formula (think this short sentence, which provides a full solution to the stated problem, out!).

Example 8.2. Consider again as the initial displacement the function that is equal to 1 on $(0.5, 1.5)$ and zero otherwise, and the initial velocity is identically zero. If the odd extensions of the given initial conditions are used then I can use the usual d'Alembert's formula. The details are given in Fig. 1.

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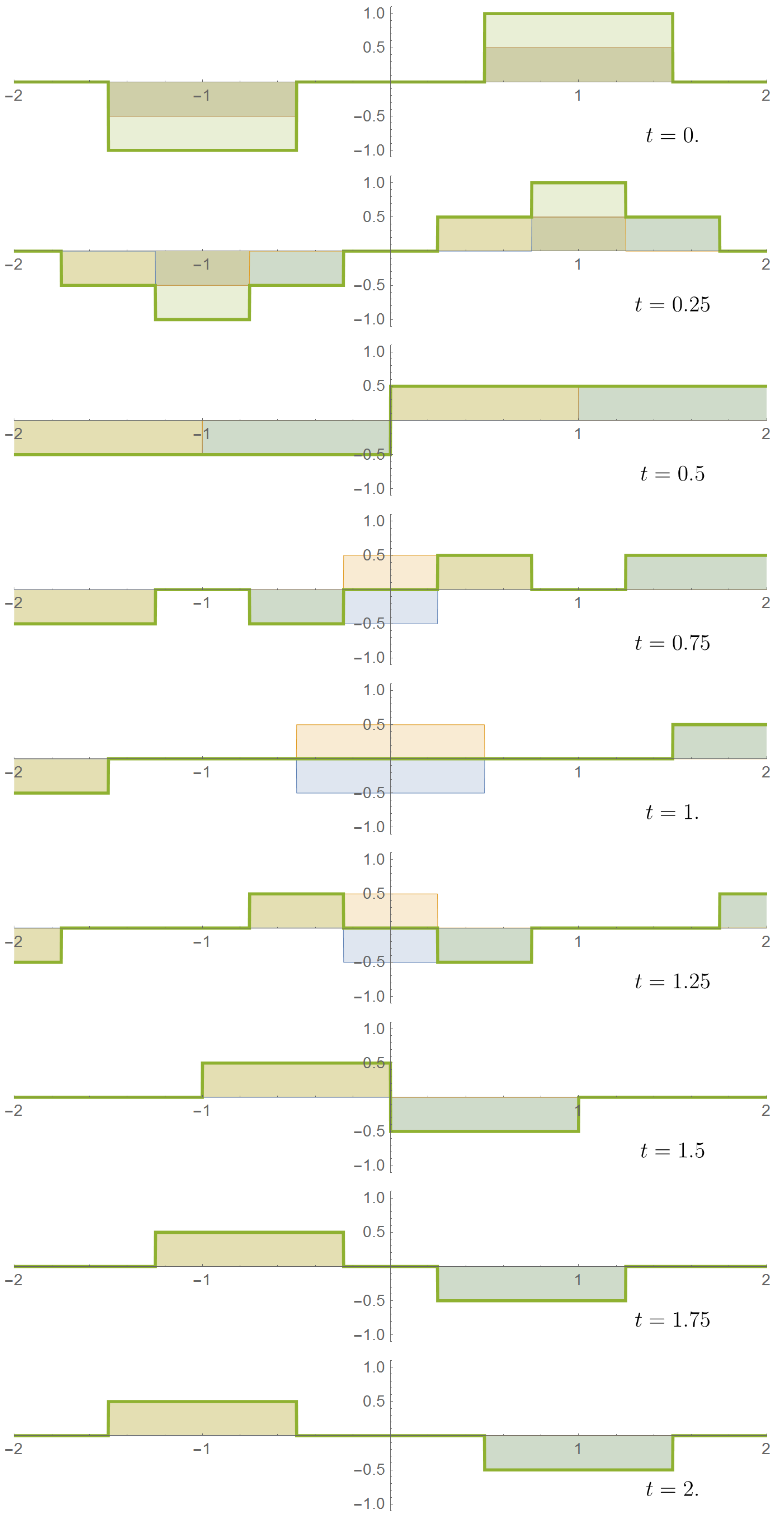

Figure 1: Solution to the wave equation with one fixed end and zero initial velocity.

I can treat similarly the case when the initial displacement is zero and the initial velocity is now given as the same box function (see Fig. 2).

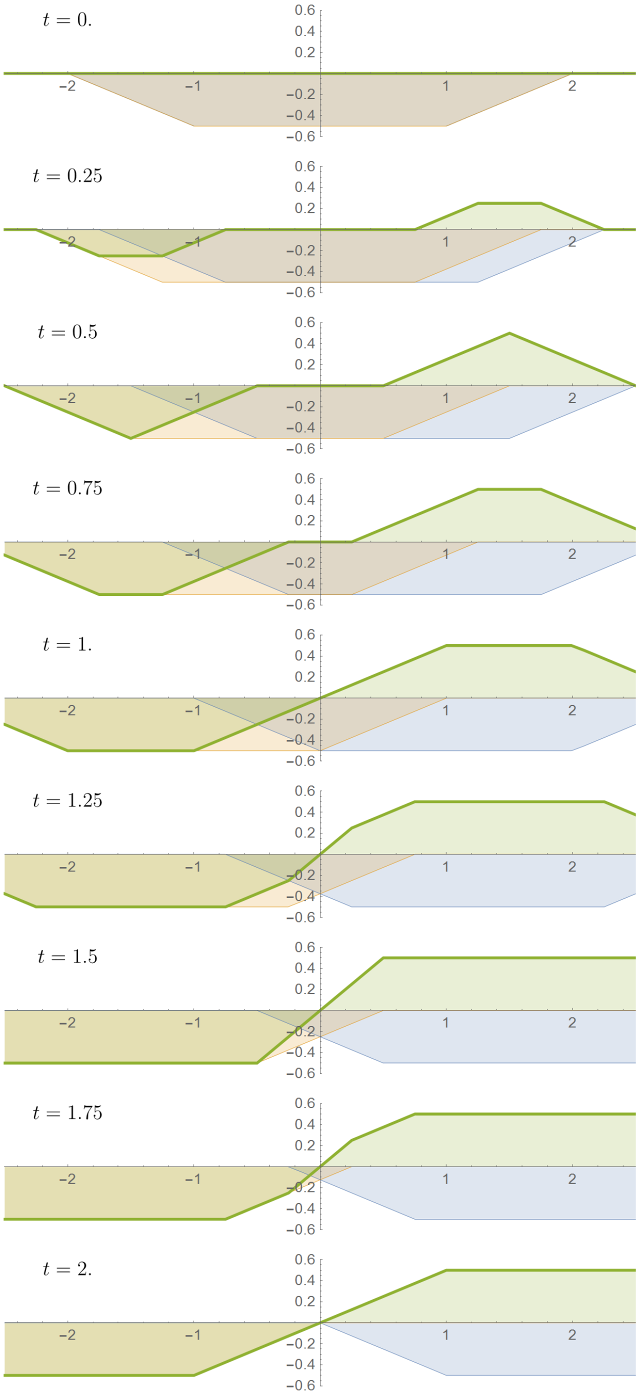

Figure 2: Solution to the wave equation with one fixed end and zero initial displacement.

How to deal with the problem with a free end? Mathematically free end of a string is modeled by the condition

$$u_x(t,0)=0,$$

which replaces the fixed end condition in the previous problem.

It turns out that such condition will be automatically fulfilled if I extend my initial conditions in the even fashion. Specifically,

Lemma 8.3. *Let the initial conditions for the wave equation on the infinite string be even functions with respect to some point x_0 then the derivative of the corresponding solution at this point is equal to zero.*

Exercise 1. Prove this lemma.

Hence to solve the problem for the wave equation with a free end one just needs to extend the initial conditions, given for $x>0$, to the whole axis in an even fashion, see Fig. 3. I will leave it as an exercise to realize what happens with the string if the initial velocity is given.

Exercise 2. Find the formulas that solve problem (8.1) explicitly.

8.2 A few words about finite string

(Before reading this section it could be useful to work through Exercise 2 or at least read the solution given at the end of this section.)

It is possible to generalize the technique I described above to the case when I have certain boundary conditions at the both ends of my string. The resulting formulas, are, however, depend heavily on the point (t,x) at which I need my solution and not very useful if I need the values of u at more than several points.

Instead of dealing with the general situation, let me consider the following problem;

$$\begin{aligned} u_{tt}&=c^2u_{xx}, \quad 0<x<l, \quad t>0,\\ u(0,x)&=x^2(1-x), \quad 0<x<l,\\ u_t(0,x)&=(1-x)^2, \quad 0<x<l,\\ u(t,0)&=u(t,l)=0, \quad t>0, \end{aligned} \tag{8.2}$$

that is, I consider the problem for the string of length l with both ends fixed.

Let me ask a specific question: Assuming that $c=l=1$, what is the value of my unknown function u at the point $(2,2/3)$?

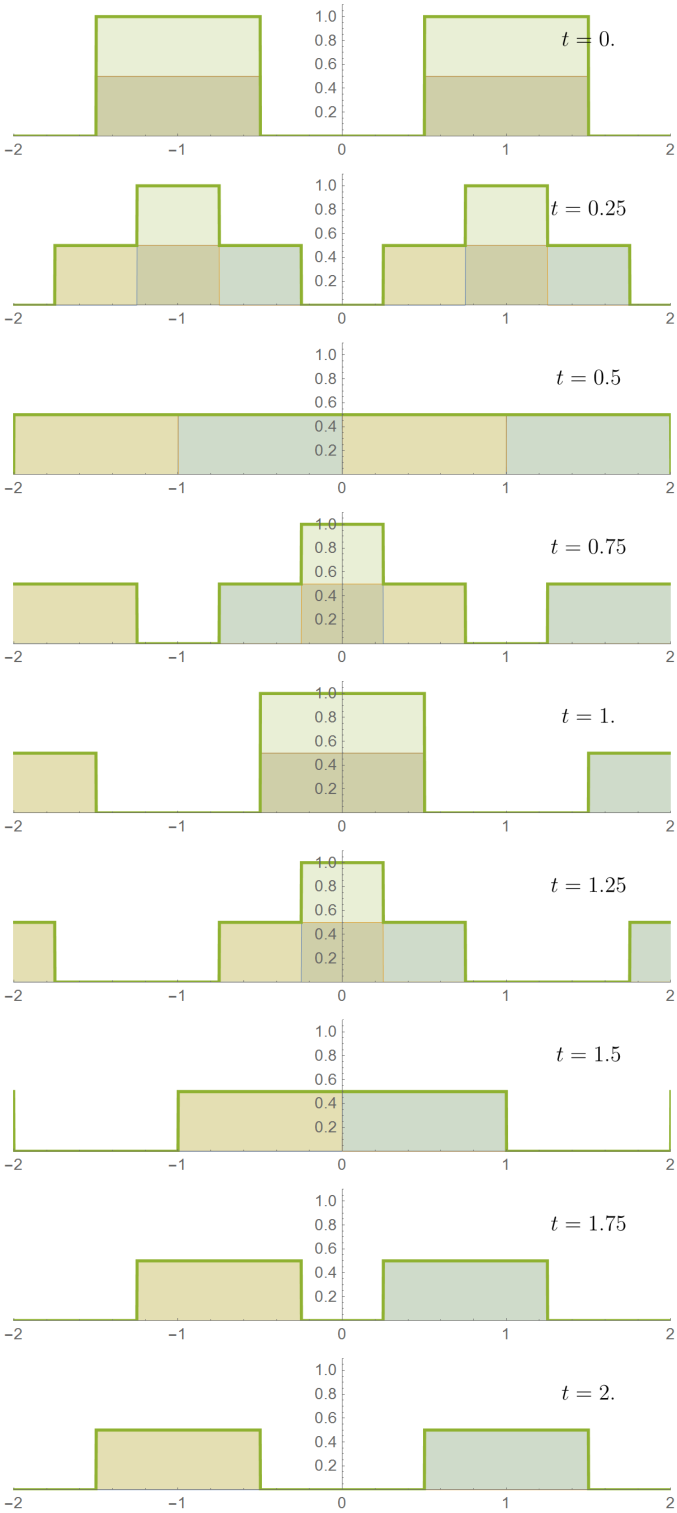

Figure 3: Solution to the wave equation with a free end at $x = 0$ and zero initial velocity.

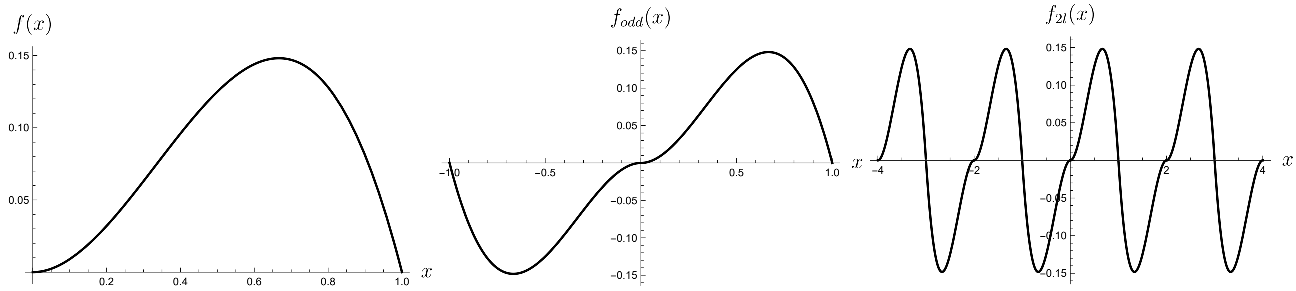

Figure 4: Extending the initial conditions from $0 < x < l$ to the whole real line. Left: the initial data on the interval $(0, l)$; center: the odd extension of the initial data to the interval $(-l, l)$; right: $2l$-periodic extension of the function from the previous step.

To use d'Alembert's formula, I need to extend my initial conditions, that are given only for $0 < x < l$, to the whole real line. I will illustrate how it is done on the example for f see Fig. 4, left panel. First, I construct an odd extension of f on the interval $(-l, l)$ (Fig. 4, central panel), and after this I replicate the obtained graph periodically, thus obtaining $2l$-periodic function (Fig. 4, right panel). The same operation should be performed with g.

In Fig. 5 one can see on the left how my $2l$-periodic extension can be expressed through the original function f. In the same figure I presented again a number of characteristics, and it is seen that the characteristics through the point $(2, 2/3)$ end up at the points $-4/6$ and $8/3$ at the line of the initial conditions (the red characteristics), hence I already can easily calculate the first part of d'Alembert's formula. For the second part, however (the one that has the integral), situation is even simpler, since (check Fig. 4, the right panel, again), the integral from $-4/3$ to $8/3$ covers exactly 2 periods of my

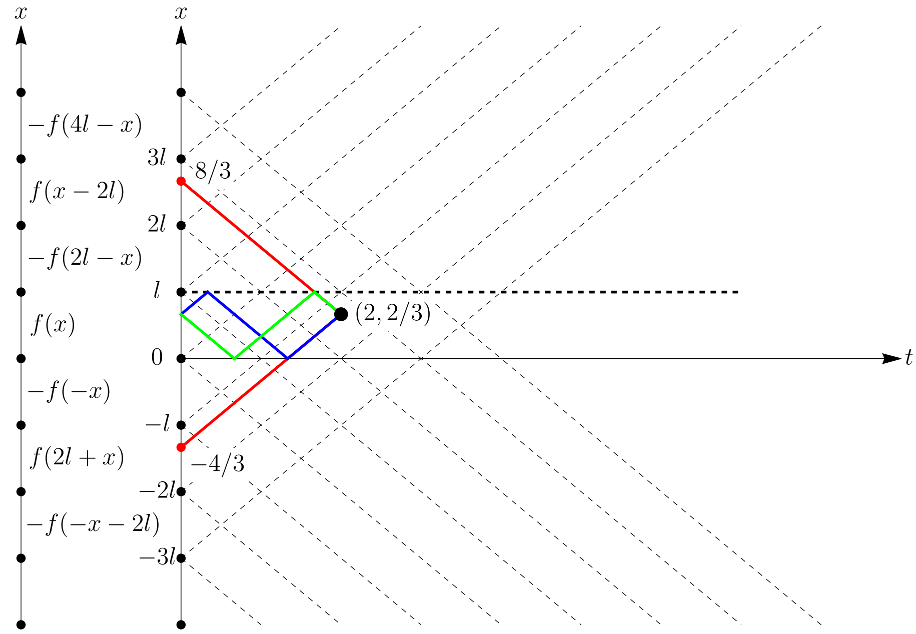

Figure 5: Computing the solution at the point $(2, 2/3)$ by the reflection method.

$2l$-periodic function, and since each period integrates to zero (it is an odd function), the contribution of g is absent.

Hence, my final answer is

$$u(2,2/3)=\frac{f_{2l}(-4/3)+f_{2l}(8/3)}{2}=\frac{f(2/3)+f(2/3)}{2}=f(2/3),$$

where the last part can be also obtained by the reflections method (see the blue and green paths in Fig. 5), moreover, the number of reflections from the boundaries of my domain, determine whether I need to use f or $-f$ and at which points. The final answer is therefore 4/27 (and not 11/81 as it is incorrectly stated in the well known textbook by W. Strauss for this problem). Later I will show that the considered problem must have 2-periodic solution, which gives the required answer without any computations.

8.3 Test yourself

8.1. Let the initial displacement has the form of a triangle and the initial velocity is zero. Sketch the solution to (8.1) at several representative time moments.

8.2. Now assume that the initial displacement is zero and the initial velocity is given by a triangular shape. Sketch the solution to (8.1) at several representative time moments.

8.3. Repeat the previous two exercises with the condition $u(t,0)=0$ (i.e., the left end is fixed) is replaced with $u_x(t,0)=0$ (the end is free).

8.4 Solutions to the exercises

Exercise 1. Again, without lost of generality I assume that $x_0=0$ (a function f is even with respect to x_0 if $f(x-x_0)=f(x_0-x)$; by making the change of variables $s=x-x_0$ I can always reduce my problem to the case $x_0=0$). Note that if f is an even function, i.e., $f(x)=f(-x)$ then its derivative is an odd function: $f'(x)=-f'(-x)$. Using this condition and Leibnitz's integral rule for d'Alembert's formula, I have, since f' is odd and g is even,

$$u_x(t,0)=\frac{f'(-ct)+f'(ct)}{2}+\frac{g(ct)-g(-ct)}{2c}=0,$$

as required. ■

Exercise 2. Let me define first the odd extensions of f and g:

$$f_{odd}(x)=\begin{cases}f(x), & x\geq 0,\\ -f(-x), & x<0,\end{cases}\qquad g_{odd}(x)=\begin{cases}g(x), & x\geq 0,\\ -g(-x), & x<0.\end{cases}$$

Now the solution to (8.1) is given, of course, as

$$u(t,x)=\frac{f_{odd}(x+ct)+f_{odd}(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}g_{odd}(s)\mathrm{d}s.$$

My goal is to rewrite this solution in terms of f and g only. I note that if the arguments of f and g are nonnegative, I can just replace them with f and g. This will always be true for $x+ct$ and for

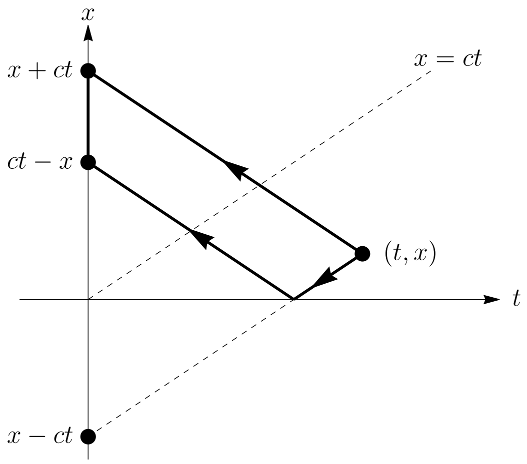

Figure 6: An illustration to the solution to Exercise 2. See text for the details.

$x-ct>0$ (recall that we are solving only for $x>0$). Now consider more interesting case $0<x<ct$, in this case $x-ct<0$ and I need to use my odd extensions, specifically,

$$u(t,x)=\frac{f(x+ct)-f(ct-x)}{2}+\frac{1}{2c}\int_{x-ct}^{0}\big(-g(-s)\big)\mathrm{d}s+\frac{1}{2c}\int_{0}^{x+ct}g(s)\mathrm{d}s,$$

which simplifies to

$$u(t,x)=\frac{f(x+ct)-f(ct-x)}{2}+\frac{1}{2c}\int_{ct-x}^{x+ct}g(s)\mathrm{d}s.$$

Hence, I showed that

$$u(t,x)=\begin{cases}\dfrac{f(x+ct)+f(x-ct)}{2}+\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct}g(s)\mathrm{d}s, & x\geq ct,\\ \dfrac{f(x+ct)-f(ct-x)}{2}+\dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct}g(s)\mathrm{d}s, & x<ct.\end{cases}$$

It may be useful to understand what happens geometrically by looking again at the plane (t,x) and characteristics (see Fig. 6). I need to determine my solution at the point (t,x) that satisfies the condition $x<ct$ (for the case $x>ct$ I use the usual d'Alembert's formula). By extending my initial conditions in the odd fashion, I can send both characteristics from (t,x) to collect the information from the initial conditions. But note that the value of my initial condition at the point $(0,x-ct)$ (which is negative) is completely determined by the value of f at the point $(0,ct-x)$, which I take with the opposite sign (because I am considering here the odd extensions). I also can integrate from $x-ct$ to $x+ct$ over my odd extension g_{odd}, but notice that the integral over symmetric interval $(x-ct,ct-x)$ is zero, and hence all I need is to integrate along $(ct-x,x+ct)$ as I derived above analytically.

In some sense (see again Fig. 6) I get the required initial data by sending my characteristics from the point (t,x) and *reflecting* the one that approaches $x=0$ to still take the data from $x>0$. Hence the name: the method of reflections. ■